

ON THE TWISTED ALEXANDER POLYNOMIAL FOR METABELIAN REPRESENTATIONS INTO $SL_2(\mathbb{C})$

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ABSTRACT. We observe the twisted Alexander polynomial for metabelian representations of knot groups into $SL_2(\mathbb{C})$ and study relations to the characterizations of metabelian representations in the character varieties. We give a factorization of the twisted Alexander polynomial for irreducible metabelian representations with the adjoint action on $\mathfrak{sl}_2(\mathbb{C})$, in which the Alexander polynomial and the twisted Alexander polynomial appear as factors. We also show several explicit examples.

1. INTRODUCTION

The purpose of this note is to expose explicit forms on the twisted Alexander polynomial of knot exteriors E_K for metabelian representations mapping the knot groups into $SL_2(\mathbb{C})$. The twisted Alexander polynomial is a refinement of the Alexander polynomial of knots by using group homomorphisms from the knot groups into linear groups as in [Lin01, Wad94]. It is of interest to consider the twisted Alexander polynomial for linear representations which send knot groups to non-abelian subgroups in $SL_2(\mathbb{C})$. We can regard irreducible metabelian representations ρ as the most simplest ones which has the non-abelian images. The author and F. Nagasato have given a characterization of metabelian representations in the $SL_2(\mathbb{C})$ -character varieties of knot groups. This characterization is summarized as

- (i) the conjugacy classes of irreducible metabelian representations form the fixed point set under an involution of the character variety of a knot group;
- (ii) every conjugacy class of an irreducible metabelian representation corresponds to a non-trivial abelian representation of the fundamental group of the double branched cover over S^3 .

From the viewpoint of a fixed point under the involution in the $SL_2(\mathbb{C})$ -character variety, there exists the induced linear isomorphism on the tangent spaces of the character variety at the fixed point. Under the identification of the twisted cohomology group and the tangent space, this linear isomorphism is given by the conjugation between irreducible metabelian representations. To see this, we will make a decomposition of $Ad \circ \rho$ into the direct sum of a 1-dimensional representation and a 2-dimensional one. The 2-dimensional direct summand is defined by another irreducible metabelian representation ${}^q\rho$. The twisted homology group defined by ${}^q\rho$ gives the tangent space with the involution.

From the decomposition of $Ad \circ \rho$, the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ turns into the product of the rational function $\Delta_K(-t)/(-t-1)$ and the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes {}^q\rho}((\sqrt{-1})t)$ where $\Delta_K(t)$ is the Alexander polynomial of K . The property that the conjugacy class of ρ is fixed by the involution deduces the symmetry of the factor $\Delta_{E_K}^{\alpha \otimes {}^q\rho}(t)$

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which is expressed as $\Delta_{E_K}^{\alpha \otimes \rho}(t) = \Delta_{E_K}^{\alpha \otimes \rho}(-t)$. Moreover we can show that $\Delta_{E_K}^{\alpha \otimes \rho}(t)$ has only even degree terms. This is derived from the property (ii) for irreducible metabelian representations and the work of P. Kirk, C. Livingston and C. Herald [HKC10].

Finally, we will touch the divisibility of the twisted Alexander polynomial for $Ad \circ \rho$. From [HKC10], if a knot K is slice, then $\Delta_{E_K}^{\alpha \otimes \rho}(t)$ is a Laurent polynomial $af(t^2)\bar{f}(t^{-2})(t^2 + 1)$ where a is a complex number and f is a Laurent polynomial with complex coefficients. In this case, the factor $t^2 + 1$ makes the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ be a Laurent polynomial. However we can see several examples for non-slice knots K and irreducible metabelian representations ρ which give that the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ are also Laurent polynomials. We give a sufficient condition on ρ for $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ to be a Laurent polynomial, which is referred as *longitude-regular*.

As summarized, we will show the following results:

Twisted Alexander polynomial with the adjoint action. *Suppose that an irreducible metabelian $SL_2(\mathbb{C})$ -representation ρ is longitude-regular. Then the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ is expressed as*

$$\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) = (t - 1) \cdot \Delta_K(-t) \cdot \frac{\Delta_{E_K}^{\alpha \otimes \rho}((\sqrt{-1})t)}{t^2 - 1}.$$

Moreover the factor $\Delta_{E_K}^{\alpha \otimes \rho}((\sqrt{-1})t)/(t^2 - 1)$ turns into a Laurent polynomial in which every term has even degree.

2. PRELIMINARIES

2.1. Review on the twisted Alexander polynomial. Let K be a knot in the 3-dimensional sphere S^3 . We denote the knot group by $\pi_1(E_K)$ where E_K is the knot exterior by removing an open tubular neighbourhood of K from S^3 . In this paper, we adopt the definition of the twisted Alexander polynomial of E_K by using Fox differential calculus, which is due to M. Wada [Wad94].

We deal with *irreducible* $SL_2(\mathbb{C})$ -representations from $\pi_1(E_K)$ into $SL_2(\mathbb{C})$ and we will use the notation ρ to denote them. Here irreducible means that $\rho(\pi_1(E_K))$ has no proper invariant line in \mathbb{C}^2 . We also consider the compositions of $SL_2(\mathbb{C})$ -representations with the adjoint action on the Lie algebra. The adjoint action is the conjugation on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ by elements in $SL_2(\mathbb{C})$:

$$Ad: SL_2(\mathbb{C}) \rightarrow \text{Aut}(\mathfrak{sl}_2(\mathbb{C}))$$

$$A \mapsto Ad_A: \mathbf{v} \mapsto A\mathbf{v}A^{-1}$$

where $\mathfrak{sl}_2(\mathbb{C})$ is regarded as a vector space over \mathbb{C} . The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by the following trace-free matrices:

$$(1) \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Remark 2.1. The adjoint action Ad_A has the eigenvalues ξ^2 , ξ^{-2} and 1 when an element A in $SL_2(\mathbb{C})$ has eigenvalues $\xi^{\pm 1}$.

2.1.1. Definition of the twisted Alexander polynomial. We review the definition of the twisted Alexander polynomial of a knot K with a representation of the knot group $\pi_1(E_K)$. It is known that every knot group $\pi_1(E_K)$ is finite presentable and expressed as follows:

$$(2) \quad \pi_1(E_K) = \langle g_1, \dots, g_k \mid r_1, \dots, r_{k-1} \rangle.$$

We choose an $\mathrm{SL}_2(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$ and denote by α the abelianization homomorphism of $\pi_1(E_K)$:

$$\alpha: \pi_1(E_K) \rightarrow \langle t \rangle = \pi_1(E_K)/[\pi_1(E_K), \pi_1(E_K)] \simeq H_1(E_K; \mathbb{Z})$$

where $\pi_1(E_K)/[\pi_1(E_K), \pi_1(E_K)]$ is regarded as a multiplicative group. Let Φ_ρ be the \mathbb{Z} -linear extension of $\alpha \otimes \rho$ on the group ring $\mathbb{Z}[\pi_1(E_K)]$ as

$$\begin{aligned} \Phi_\rho: \mathbb{Z}[\pi_1(E_K)] &\rightarrow M_2(\mathbb{C}[t^{\pm 1}]) \\ \sum_i a_i \gamma_i &\mapsto \sum_i a_i \alpha(\gamma_i) \otimes \rho(\gamma_i) \end{aligned}$$

where $M_2(\mathbb{C}[t^{\pm 1}]) = \mathbb{C}[t^{\pm 1}] \otimes_{\mathbb{C}} M_2(\mathbb{C})$ and we identify $M_2(\mathbb{C}[t^{\pm 1}])$ with the set of matrices whose entries are elements in $\mathbb{C}[t^{\pm 1}]$. We also denote by $\Phi_{Ad \circ \rho}$ the \mathbb{Z} -linear extension of $\alpha \otimes Ad \circ \rho$.

Definition 2.2. Let K be a knot and the knot group $\pi_1(E_K)$ be represented as in (2). We suppose that $\alpha(g_l) \neq 1$. Then for an $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_K)$, we define the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \rho}(t)$ as

$$(3) \quad \Delta_{E_K}^{\alpha \otimes \rho}(t) = \frac{\det \left(\Phi_\rho \left(\frac{\partial r_i}{\partial g_j} \right) \right)_{\substack{1 \leq i \leq k-1, \\ 1 \leq j \leq k, j \neq l}}}{\det(\Phi_\rho(g_l - 1))}$$

where the $\partial r_i / \partial g_j$ is a linear combination of words in g_1, \dots, g_k , given by Fox differential of r_i by g_j . We also define the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ for the composition $Ad \circ \rho$ by $\Phi_{Ad \circ \rho}$ up to a factor $\pm t^m$ where $m \in \mathbb{Z}$.

Remark 2.3. The assumption that $\alpha(g_l) \neq 1$ is a sufficient condition for the denominator to be non-zero.

Remark 2.4. The twisted Alexander polynomial of K for an $\mathrm{SL}_2(\mathbb{C})$ -representation of $\pi_1(E_K)$ does not depend on the choices of a presentation of $\pi_1(E_K)$. Moreover the twisted Alexander polynomial has the invariance under the conjugation of representations.

We refer to [Kit96, Wad94] for the details on the well-definedness. We choose the last generator as g_l in Definition 2.2 for our explicit examples in Section 5.

2.2. Review on metabelian representations of knot groups. We are interested in irreducible *metabelian* representations. This Subsection reviews briefly known results about metabelian representations.

Definition 2.5. A representation ρ is *metabelian* if the image by ρ of the commutator subgroup $[\pi_1(E_K), \pi_1(E_K)]$ is an abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$.

Remark 2.6. It is known that every maximal abelian subgroup in $\mathrm{SL}_2(\mathbb{C})$ is conjugate to either the abelian subgroup consisting of hyperbolic elements or parabolic ones, i.e.,

$$\text{Hyp} := \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \middle| z \in \mathbb{C} \setminus \{0\} \right\} \quad \text{or} \quad \text{Para} := \left\{ \begin{pmatrix} \pm 1 & w \\ 0 & \pm 1 \end{pmatrix} \middle| w \in \mathbb{C} \right\}.$$

X-S. Lin [Lin01] introduced a useful presentation of knot groups to express irreducible metabelian representations (see Definition 2.7), which is referred as a *Lin presentation*. Using a Lin presentation, we will see that the composition $Ad \circ \rho$ contains another metabelian representation ρ' as a direct summand in Proposition 3.3.

Definition 2.7 (Lemma 2.1 in [Lin01]). We suppose that a free Seifert surface S has genus g . Let $\pi_1(S^3 \setminus N(S))$ be generated by x_1, \dots, x_{2g} where each x_i corresponds to the core of 1-handle in the handlebody $S^3 \setminus N(S)$. We denote by μ a meridian on ∂E_K . Then the knot group $\pi_1(E_K)$ is expressed as

$$(4) \quad \pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$$

where α_i and β_i are words in x_1, \dots, x_{2g} obtained by pushing up and pushing down the spine $\vee_{i=0}^{2g} a_i$ of S along the normal direction in $N(S)$. We call the presentation in (4) the Lin presentation associated with S .

For example, Figure 1 shows a free Seifert surface of the left handed trefoil knot. The closed loops x_1 and x_2 correspond to the core of 1-handles in the complement of this Seifert surface. The spine $\vee_{i=1}^2 a_i$ is illustrated in Figure 1. It is easy to see that a_1^+, a_1^-, a_2^+ and a_2^- are homotopic to $x_1, x_1 x_2^{-1}, x_2^{-1} x_1$ and x_2^{-1} .

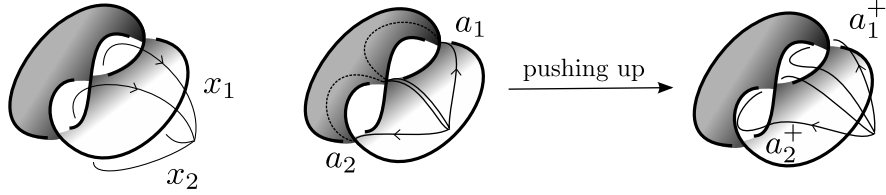


FIGURE 1. The spine of the free Seifert surface

We see explicit forms of irreducible metabelian representation via a Lin presentation of $\pi_1(E_K)$, which is due to Lin [Lin01] and F. Nagasato [Nag07].

Proposition 2.8 (Proposition 1.1 in [Nag07]). *We choose that a Lin presentation of $\pi_1(E_K)$ as*

$$\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle.$$

Then every irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation ρ is conjugate to one given by the following correspondences:

$$(5) \quad x_i \mapsto \begin{pmatrix} z_i & 0 \\ 0 & z_i^{-1} \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where each z_i is a root of unity whose order is a divisor of $|\Delta_K(-1)|$.

In the set of irreducible $\mathrm{SL}_2(\mathbb{C})$ -metabelian representations, metabelian representations are characterized as follows.

Proposition 2.9 (Proposition 3 and Theorem 1 in [NY12]). *The number of conjugacy classes of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representations is given by $(|\Delta_K(-1)| - 1)/2$. These conjugacy classes forms the fixed point set in the set of conjugacy classes of irreducible $\mathrm{SL}_2(\mathbb{C})$ -representations ρ under the involution induced by the following correspondence:*

$$\rho \mapsto (-1)^{[\cdot]} \rho$$

where $(-1)^{[\cdot]} \in \{\pm 1\}$ and $[\gamma]$ denotes the homology class in $H_1(E_K; \mathbb{Z}) \simeq \mathbb{Z}$ for any $\gamma \in \pi_1(E_K)$.

The conjugacy class of ρ is one-to-one corresponding to the conjugacy class of an abelian representation for the double branched cover over S^3 .

3. EXPLICIT FORMS OF THE COMPOSITION $Ad \circ \rho$ FOR METABELIAN REPRESENTATIONS

We start with an explicit form of the composition of an irreducible metabelian representation with the adjoint action. From a Lin presentation of $\pi_1(E_K)$, the composition with the adjoint action can be decomposed into the direct sum of a 1-dimensional representation and a 2-dimensional one.

Proposition 3.1. *Let ρ be an irreducible metabelian representation of $\pi_1(E_K)$ into $SL_2(\mathbb{C})$, as expressed by using the correspondence (5) in Proposition 2.8. Then we have the following decomposition:*

$$Ad \circ \rho = \psi_1 \oplus \psi_2, \quad \psi_1 \in GL(V_1), \quad \psi_2 \in GL(V_2)$$

where V_1 and V_2 are the subspace $\langle H \rangle$ and $\langle E, F \rangle$ in $\mathfrak{sl}_2(\mathbb{C})$. Moreover the representations ψ_1 and ψ_2 are expressed, by taking conjugation if necessary, as follows:

$$\begin{aligned} \psi_1(x_i) &= 1, \quad \psi_1(\mu) = -1, \\ \psi_2(x_i) &= \begin{pmatrix} z_i^2 & 0 \\ 0 & z_i^{-2} \end{pmatrix}, \quad \psi_2(\mu) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

for generators of a Lin presentation in Proposition 2.8.

Proof. We can assume, if necessary by taking conjugation, that irreducible metabelian representation ρ is expressed as

$$\rho(x_i) = \begin{pmatrix} z_i & 0 \\ 0 & z_i^{-1} \end{pmatrix}, \quad \rho(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for a Lin presentation $\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$. By direct calculation, the composition $Ad \circ \rho$ is expressed as

$$Ad \circ \rho(x_i) = \begin{pmatrix} z_i^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z_i^{-2} \end{pmatrix}, \quad Ad \circ \rho(\mu) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

with respect to the basis $\{E, H, F\}$ of $\mathfrak{sl}_2(\mathbb{C})$ as in (1). \square

Remark 3.2. An irreducible metabelian representation ρ as in Proposition 2.8 is conjugate to $(-1)^{[\cdot]} \rho$ by the matrix $C = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. The conjugation of C acts on V_2 as $-\text{id}$ and on V_1 as id .

The 2-dimensional representation ψ_2 is related to another irreducible metabelian representation into $SL_2(\mathbb{C})$. By taking conjugation of ψ_2 by the matrix

$$D = \begin{pmatrix} e^{3\pi\sqrt{-1}/4} & 0 \\ 0 & e^{-3\pi\sqrt{-1}/4} \end{pmatrix},$$

we can see that this conjugate representation gives the following correspondence:

$$(6) \quad \mu \mapsto \sqrt{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x_i \mapsto \begin{pmatrix} z_i^2 & 0 \\ 0 & z_i^{-2} \end{pmatrix}$$

for a Lin presentation $\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$. From the correspondence (6) and Proposition 2.8, we can see that $D\psi_2 D^{-1}$ is expressed as $(\sqrt{-1})^{[\cdot]} \rho'$ where ρ' is another irreducible metabelian $SL_2(\mathbb{C})$ -representation of $\pi_1(E_K)$ (see Proposition 2.9 for the notation $(\sqrt{-1})^{[\cdot]}$). Hence we have shown the following embedding of irreducible metabelian representation ρ' into the composition of another irreducible metabelian representation and the adjoint action.

Proposition 3.3. *Let ρ be an irreducible metabelian representation of $\pi_1(E_K)$ into $\mathrm{SL}_2(\mathbb{C})$. There exists an irreducible metabelian representation ρ' such that $\mathrm{Ad} \circ \rho$ is conjugate to the direct sum $(-1)^{[\cdot, 1]} \oplus (\sqrt{-1})^{[\cdot, 1]} \rho'$ as an $\mathrm{SL}_3(\mathbb{C})$ -representation.*

We denote by ${}^q\rho$ this $\mathrm{SL}_2(\mathbb{C})$ -representation ρ' associated with ρ .

4. COMPUTATION OF THE TWISTED ALEXANDER POLYNOMIAL

This section shows several computation results for irreducible metabelian representations. The purpose of this section is to provide an explicit form of the twisted Alexander polynomial for the composition with the adjoint action. Moreover we will see that a relation of our result to the twisted Alexander polynomial for the standard action of $\mathrm{SL}_2(\mathbb{C})$.

4.1. Computation for irreducible metabelian representations with and without the adjoint action. We start with the twisted Alexander polynomial without the adjoint action, which appears as a factor in $\Delta_{E_K}^{\alpha \otimes \mathrm{Ad} \circ \rho}(t)$.

Proposition 4.1. *For any irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation ρ' , the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \rho'}(t)$ is a Laurent polynomial which consists of only even degree terms in t .*

Proof. We choose a Lin presentation of $\pi_1(E_K)$:

$$\langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle$$

and suppose that the representation ρ' sends the generators to the following matrices:

$$\rho'(x_i) = \begin{pmatrix} z_i & 0 \\ 0 & z_i^{-1} \end{pmatrix}, \quad \rho'(\mu) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since some commutator x_i satisfies that $\rho'(x_i) \neq \mathbf{1}$, the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \rho'}(t)$ turns into a Laurent polynomial by Wada's criterion [Wad94, Proposition 8]. From Proposition 2.9, the conjugacy class of ρ' corresponds to that of an abelian representation for the double branched cover. Our invariant coincides with the twisted Alexander polynomial replaced the variable t with $-t^2$ in [HKC10, Theorem 7.1] (we refer to the formula (7.4) in [HKC10] for changing variables). \square

Next we consider the twisted Alexander polynomial for the composition of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representations with the adjoint action. We assume a technical condition called “longitude-regular” for $\mathrm{SL}_2(\mathbb{C})$ -representations. This condition guarantees the twisted Alexander polynomial for the composition of $\mathrm{SL}_2(\mathbb{C})$ -representation with the adjoint action to be a Laurent polynomial.

Theorem 4.2. *Let an $\mathrm{SL}_2(\mathbb{C})$ -representation ρ be longitude-regular and metabelian. Then the twisted Alexander polynomial for the composition of ρ with the adjoint action is expressed as*

$$\Delta_{E_K}^{\alpha \otimes \mathrm{Ad} \circ \rho}(t) = (t - 1) \Delta_K(-t) P(t)$$

where $P(t)$ is a Laurent polynomial satisfying that $P(t) = P(-t)$.

Remark 4.3. The assumption that ρ is longitude-regular includes the irreducibility of ρ (see Definition 4.10).

Proof. We choose a Lin presentation of $\pi_1(E_K)$ as

$$\pi_1(E_K) = \langle x_1, \dots, x_{2g}, \mu \mid \mu a_i^+ \mu^{-1} = a_i^-, i = 1, \dots, 2g \rangle.$$

By Proposition 3.1, we can assume that $Ad \circ \rho = \psi_2 \oplus \psi_1$ such that

$$\begin{aligned} \psi_1(x_i) &= 1, \quad \psi_1(\mu) = -1, \\ \psi_2(x_i) &= \begin{pmatrix} z_i^2 & 0 \\ 0 & z_i^{-2} \end{pmatrix}, \quad \psi_2(\mu) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

The twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ for the composition $Ad \circ \rho$ is factored into the product of $\Delta_{E_K}^{\alpha \otimes \psi_1}(t)$ and $\Delta_{E_K}^{\alpha \otimes \psi_2}(t)$. From [KL99], it is known that the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \psi_1}(t)$ is expressed as the rational function $\Delta_K(-t)/(-t-1)$. We can see that the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \psi_2}(t)$ turns into a Laurent polynomial by Wada's criterion [Wad94, Proposition 8] since there exists a commutator x_i such that $\psi_2(x_i) \neq 1$. Moreover ψ_2 is conjugate to $(-1)^{[-1]} \psi_2$ by the matrix C in Remark 3.2, which implies that the Laurent polynomial $\Delta_{E_K}^{\alpha \otimes \psi_2}(t)$ has the symmetry that $\Delta_{E_K}^{\alpha \otimes \psi_2}(t) = \Delta_{E_K}^{\alpha \otimes \psi_2}(-t)$. Summarizing the above, we have

$$\begin{aligned} \Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) &= \Delta_{E_K}^{\alpha \otimes \psi_1}(t) \cdot \Delta_{E_K}^{\alpha \otimes \psi_2}(t) \\ (7) \quad &= \frac{\Delta_K(-t)}{-t-1} \cdot Q(t) \end{aligned}$$

where $Q(t)$ is a Laurent polynomial satisfying that $Q(t) = Q(-t)$.

Since ρ is longitude-regular, it follows from [Yam08] that the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t)$ has zero at $t = 1$. It is known that $\Delta_K(-1)$ is an odd integer. Hence the Laurent polynomial $Q(t)$ has zero at $t = 1$. Together with the symmetry that $Q(t) = Q(-t)$, we can factor $Q(t)$ into the product $(t-1)(t+1)P(t)$. This factorization of $Q(t)$ completes the proof when substituted in (7). \square

The factorization of Theorem 4.2 is deduced from the decomposition $Ad \circ \rho = \psi_1 \oplus \psi_2$ for an irreducible metabelian representation ρ in Proposition 3.1. Furthermore Proposition 3.3 shows that the polynomial $P(t)$ is given by the twisted Alexander polynomial for ${}^q\rho$.

Theorem 4.4. *For an irreducible metabelian $SL_2(\mathbb{C})$ -representation ρ , let ψ_2 be the 2-dimensional direct summand in the composition $Ad \circ \rho$. Then we have the following equation of the twisted Alexander polynomials:*

$$\Delta_{E_K}^{\alpha \otimes \psi_2}(t) = \Delta_{E_K}^{\alpha \otimes {}^q\rho}((\sqrt{-1})t)$$

And the factor $P(t)$ in Theorem 4.2 is given by $\Delta_{E_K}^{\alpha \otimes {}^q\rho}((\sqrt{-1})t)/(t^2-1)$, i.e.,

$$\Delta_{E_K}^{\alpha \otimes Ad \circ \rho}(t) = (t-1) \cdot \Delta_K(-t) \cdot \frac{\Delta_{E_K}^{\alpha \otimes {}^q\rho}((\sqrt{-1})t)}{t^2-1},$$

where $\Delta_{E_K}^{\alpha \otimes {}^q\rho}((\sqrt{-1})t)/(t^2-1)$ is a Laurent polynomial which consists of only even degree terms.

Proof. It follows from $\Delta_{E_K}^{\alpha \otimes \psi_2}(t) = \Delta_{E_K}^{\alpha \otimes (\sqrt{-1})^{[-1]} \rho}(t) = \Delta_{E_K}^{(\sqrt{-1})^{[-1]} \alpha \otimes \rho}(t) = \Delta_{E_K}^{\alpha \otimes {}^q\rho}((\sqrt{-1})t)$ and $P(t)$ is $\Delta_{E_K}^{\alpha \otimes \psi_2}(t)/(t^2-1)$. Proposition 4.1 implies that every term in $\Delta_{E_K}^{\alpha \otimes \psi_2}(t)$ has even degree. \square

4.2. On the twisted homology group and the longitude-regularity. In this Subsection, we will touch the detail on the twisted homology group and the longitude-regularity of irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representations. The longitude-regularity of an $\mathrm{SL}_2(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$ consists of the following conditions on the twisted chain complex $C_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ given by $Ad \circ \rho$.

The twisted chain complex $C_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is defined as follows. We denote by \widetilde{E}_K the universal cover of E_K . The chain complex $C_*(\widetilde{E}_K; \mathbb{Z})$ consists of left $\mathbb{Z}[\pi_1(E_K)]$ -modules via covering transformation by $\pi_1(E_K)$. Under the action of $\pi_1(E_K)$ by $Ad \circ \rho^{-1}$, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is a right $\mathbb{Z}[\pi_1(E_K)]$ -module.

Definition 4.5. By taking tensor product of $C_*(\widetilde{E}_K; \mathbb{Z})$ with $\mathfrak{sl}_2(\mathbb{C})$, we have the chain complex $\mathfrak{sl}_2(\mathbb{C}) \otimes_{Ad \circ \rho^{-1}} C_*(\widetilde{E}_K; \mathbb{Z})$. This local system is called the twisted chain complex of E_K with the coefficient $\mathfrak{sl}_2(\mathbb{C})_\rho$ and denoted by $C_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$. We denote by $H_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ the homology group of $C_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$.

A special case in the result [BF11] of H. Boden and S. Friedl have shown the dimension of $H_i(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ as follows.

Proposition 4.6 ([BF11]). *For any irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation ρ , we have $\dim_{\mathbb{C}} H_1(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = 1$. Moreover from the irreducibility of ρ and Poincaré duality, we can see the dimension of $H_i(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ for all i as follows:*

$$\dim_{\mathbb{C}} H_i(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho) = \begin{cases} 1 & i = 1 \text{ or } 2, \\ 0 & \text{otherwise.} \end{cases}$$

We study the homology group $H_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ under the involution in Proposition 2.9. By Proposition 3.1, The chain complex $C_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is decomposed into the direct sum of $V_1 \otimes_{\psi_1^{-1}} C_*(\widetilde{E}_K; \mathbb{Z})$ and $V_2 \otimes_{\psi_2^{-1}} C_*(\widetilde{E}_K; \mathbb{Z})$ when an irreducible $\mathrm{SL}_2(\mathbb{C})$ -representation ρ is metabelian. We denote by $C_*(E_K; V_i)$ the subchain complex $V_i \otimes_{\psi_i^{-1}} C_*(\widetilde{E}_K; \mathbb{Z})$ for $i = 1, 2$ and use the notation $H_*(E_K; V_i)$ for the homology group.

Proposition 4.7. *If $\mathrm{SL}_2(\mathbb{C})$ -representation ρ is irreducible metabelian, then the homology group $H_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$ is isomorphic to the homology group $H_*(E_K; V_2)$.*

Proof. By the decomposition of the chain complex, we have the decomposition of the homology group:

$$H_*(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho) \simeq H_*(E_K; V_1) \oplus H_*(E_K; V_2)$$

The proof follows from the next Lemma 4.8. □

Lemma 4.8. *The homology group $H_*(E_K; V_1)$ is trivial.*

Proof. The representation ψ_1 is the non-trivial 1-dimensional representation such that a meridian μ is sent to -1 . Our claim follows from the computation of example 2 in [KL99] and $\Delta_K(-1) \neq 0$. □

Remark 4.9. The conjugation between ρ and $(-1)^{[1]} \rho$ induces the linear isomorphism $-\mathrm{id}$ on $H_1(E_K; V_2)$ (see Remark 3.2).

The longitude-regularity of ρ is defined concerning a basis of $H_1(E_K; \mathfrak{sl}_2(\mathbb{C})_\rho)$. It was introduced by J. Porti in [Por97]. Here we follows the definition in [Dub05, Yam08].

Definition 4.10. An $\mathrm{SL}_2(\mathbb{C})$ -representation ρ is longitude-regular if it is irreducible and satisfies the following two conditions on the local system with the coefficient $\mathfrak{sl}_2(\mathbb{C})_\rho$:

- (i) $\dim_{\mathbb{C}} H_1(E_K; \mathfrak{sl}_2(\mathbb{C})_{\rho}) = 1$,
- (ii) the homomorphism $H_1(\lambda; \mathfrak{sl}_2(\mathbb{C})_{\rho}) \rightarrow H_1(E_K; \mathfrak{sl}_2(\mathbb{C})_{\rho})$, induced by the inclusion $\lambda \rightarrow E_K$, is surjective. Here λ is the preferred longitude on ∂E_K and $H_1(\lambda; \mathfrak{sl}_2(\mathbb{C})_{\rho})$ is the homology group for the local system of the circle λ , given by restricting ρ on the subgroup $\langle \lambda \rangle \subset \pi_1(E_K)$.
- (iii) If $\rho(\pi_1(\partial E_K))$ is contained in Para by conjugation, then $\rho(\lambda) \neq \pm \mathbf{1}$.

The conjugation preserves the longitude regularity of $SL_2(\mathbb{C})$ -representations. Proposition 2.8 shows that every irreducible metabelian $SL_2(\mathbb{C})$ -representation sends meridians to trace free matrices. Thus we ignore the third condition in Definition 4.10 for irreducible metabelian representations. Proposition 4.6 shows that all irreducible metabelian representations satisfy the condition on the dimension of local system in Definition 4.10.

We give more details on the induced homomorphism from $H_1(\lambda; \mathfrak{sl}_2(\mathbb{C})_{\rho})$ to the twisted homology group $H_1(E_K; \mathfrak{sl}_2(\mathbb{C})_{\rho})$. It is known that the homotopy class of the preferred longitude λ is contained in the second commutator subgroup of $\pi_1(E_K)$. If an $SL_2(\mathbb{C})$ -representation ρ of $\pi_1(E_K)$ is metabelian, the preferred longitude λ is sent to the identity matrix $\mathbf{1}$ by ρ . Hence we can identify the local system $C_*(\lambda; \mathfrak{sl}_2(\mathbb{C})_{\rho})$ with $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{Z}} C_*(\lambda; \mathbb{Z})$. It is easy to see the twisted homology group for λ .

Lemma 4.11. *If an irreducible metabelian representation of $\pi_1(E_K)$ into $SL_2(\mathbb{C})$, then the homology group of $C_*(\lambda; \mathfrak{sl}_2(\mathbb{C})_{\rho})$ is expressed as*

$$H_*(\lambda; \mathfrak{sl}_2(\mathbb{C})_{\rho}) = \begin{cases} \mathfrak{sl}_2(\mathbb{C}) & * = 0 \text{ or } 1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Remark 4.12. This Lemma 4.11 says that an irreducible metabelian representation ρ is longitude-regular if and only if there exists some $\mathbf{v} \in \mathfrak{sl}_2(\mathbb{C})$ such that $\mathbf{v} \otimes \lambda$ gives a non-trivial element in $H_1(E_K; \mathfrak{sl}_2(\mathbb{C})_{\rho})$. Here we use the same symbol λ for a lift in a universal cover for simplicity of notation.

By the long exact sequence for the pair $(E_K, \partial E_K)$ and Proposition 4.6 & 4.7, we have the following short exact sequence of homology groups with coefficient V_2 :

$$0 \rightarrow H_2(E_K, \partial E_K; V_2) \xrightarrow{\delta_*} H_1(\partial E_K; V_2) \xrightarrow{i_*} H_1(E_K; V_2) \rightarrow 0$$

and

$$\dim_{\mathbb{C}} \ker i_* = 1, \quad \dim_{\mathbb{C}} \operatorname{Im} i_* = 1.$$

By direct calculation, we have a basis of $H_1(\partial E_K; V_2)$ as follows:

Lemma 4.13. *Let \mathbf{v}_1 be an eigenvector $E - F$ in $V_2 = \langle E, F \rangle$ for the eigenvalue 1 of $\psi_2(\mu)$. The homology group $H_1(\partial E_K; V_2)$ is generated by the homology classes of $\mathbf{v}_1 \otimes \lambda$ and $\mathbf{v}_1 \otimes \mu$. Here we denote by the same symbols λ and μ a lifts in the universal cover.*

We can write $a[\mathbf{v}_1 \otimes \lambda] + b[\mathbf{v}_1 \otimes \mu]$ for a basis of $\ker i_*$. This pair of complex numbers determines the point $[a : b]$ in the projective space \mathbb{CP}^1 . From Remark 4.12, it follows that an irreducible metabelian $SL_2(\mathbb{C})$ -representation ρ is longitude-regular if and only if $[a : b] \neq [1 : 0]$. It is sufficient to consider the condition for the ratio b/a to be non-zero.

The ratio b/a can be expressed as a ratio of 1-forms on the set of conjugacy classes of irreducible representations. For each $\gamma \in \pi_1(E_K)$, we define the function I_{γ} on $SL_2(\mathbb{C})$ -representations as $I_{\gamma}(\rho) = \operatorname{tr} \rho(\gamma)$ where ρ is an $SL_2(\mathbb{C})$ -representation. By the invariance of trace under conjugation, this function I_{γ} also gives a function on the set of conjugacy classes of $SL_2(\mathbb{C})$ -representations. Then we have the following condition for an irreducible

metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation to be longitude-regular. We refer to [Por97, Proposition 4.7] and its proof for the details.

Proposition 4.14. *Suppose that the function I_λ is determined by I_μ near the conjugacy class of an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation ρ . The ratio b/a is expressed as*

$$(8) \quad \left(\frac{b}{a}\right)^2 = \frac{I_\mu^2 - 4}{I_\lambda^2 - 4} \left(\frac{dI_\lambda}{dI_\mu}\right)^2$$

Therefore b/a does not vanish at $(I_\lambda, I_\mu) = (2, 0)$ if and only if an irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation ρ is longitude-regular.

For example, we can see the relation $I_\lambda = -(I_\mu^6 - 6I_\mu^4 + 9I_\mu^2 - 2)$ in the case that K is the trefoil knot. The right hand side of (8) turns into 36. This means that the ratio b/a is not zero at $(I_\lambda, I_\mu) = (2, 0)$. Hence every irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of the trefoil knot group is longitude-regular. In the case that K is the figure eight knot, one can see the relation $I_\lambda = I_\mu^4 - 5I_\mu^2 + 2$ (we refer to [Por97, Section 4.5, Example 1]). The right hand side of (8) turns into $4(2I_\mu^2 - 5)^2 / (I_\mu^2 - 1)(I_\mu^2 - 5)$. This means that the ratio b/a is not zero at $(I_\lambda, I_\mu) = (2, 0)$. Hence every irreducible metabelian $\mathrm{SL}_2(\mathbb{C})$ -representation of the figure eight knot group is longitude-regular.

5. EXAMPLES

We show four computation examples along Rolfsen's table of knots. The first two examples are the trefoil knot and the figure eight knot, for which the extra factor $P(t)$ in Theorem 4.2 is trivial. We can see the non-trivial $P(t)$ in the two examples for 5_1 and 5_2 knot. In each example, the representation ${}^q\rho_i$ associated with ρ_i is given by ρ_{i+1} cyclically.

5.1. Trefoil knot. Let K be the trefoil knot. We consider the free Seifert surface illustrated as in Figure 1. The Lin presentation associated to this Seifert surface is expressed as

$$(9) \quad \pi_1(E_K) = \langle x_1, x_2, \mu \mid \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \mu x_2^{-1} x_1 \mu^{-1} = x_2^{-1} \rangle.$$

The Alexander polynomial $\Delta_K(t)$ is given by $t^2 - t + 1$. The knot determinant is $\Delta_K(-1) = 3$. Since the number of conjugacy classes of irreducible metabelian representations is given by $(|\Delta_K(-1)| - 1)/2$, we have only one irreducible metabelian representation, up to conjugate. By Proposition 4.1 and the relations in the Lin presentation (9), we have the following representative in the conjugacy class of the irreducible metabelian representation:

$$\rho : x_1 \mapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} \zeta_3^2 & 0 \\ 0 & \zeta_3^{-2} \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\zeta_3 = \exp(2\pi\sqrt{-1}/3)$. With that only the generator μ satisfies that $\alpha(\mu) \neq 1$ in mind, the twisted Alexander polynomials $\Delta_{E_K}^{\alpha \otimes \rho}(t)$ and $\Delta_{E_K}^{\alpha \otimes \mathrm{Ad} \circ \rho}(t)$ are expressed as

$$\Delta_{E_K}^{\alpha \otimes \rho}(t) = \frac{\det\left(\Phi_\rho\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1 \leq i \leq 2, \\ 1 \leq j \leq 2}}}{\det(\Phi_\rho(\mu - 1))}, \quad \Delta_{E_K}^{\alpha \otimes \mathrm{Ad} \circ \rho}(t) = \frac{\det\left(\Phi_{\mathrm{Ad} \circ \rho}\left(\frac{\partial r_i}{\partial x_j}\right)\right)_{\substack{1 \leq i \leq 2, \\ 1 \leq j \leq 2}}}{\det(\Phi_{\mathrm{Ad} \circ \rho}(\mu - 1))}$$

where $r_1 = \mu x_1 \mu^{-1} x_2 x_1^{-1}$ and $r_2 = \mu x_2^{-1} x_1 \mu^{-1} x_2$. Then we have, up to a factor $\pm t^n$ ($n \in \mathbb{Z}$),

$$\begin{aligned} \Delta_{E_K}^{\alpha \otimes \rho}(t) &= t^2 + 1, \\ \Delta_{E_K}^{\alpha \otimes \mathrm{Ad} \circ \rho}(t) &= (t - 1)(t^2 + t + 1) \\ &= (t - 1)\Delta_K(-t). \end{aligned}$$

5.2. Figure eight knot. Let K be the figure eight knot. We consider the free Seifert surface illustrated as in Figure 2. The Lin presentation associated to this Seifert surface is expressed as

$$(10) \quad \pi_1(E_K) = \langle x_1, x_2, \mu \mid \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \mu x_2 x_1 \mu^{-1} = x_2 \rangle.$$



FIGURE 2. A free Seifert surface of the figure eight knot

The Alexander polynomial $\Delta_K(t)$ is given by $t^2 - 3t + 1$. The knot determinant is $\Delta_K(-1) = 5$. Since the number of conjugacy classes of irreducible metabelian representations is given by $(|\Delta_K(-1)| - 1)/2$, we have two irreducible metabelian representations, up to conjugate. By Proposition 2.8 and the relations in the Lin presentation (10), we have the following representatives ρ_1 and ρ_2 in the two conjugacy classes of irreducible metabelian representations:

$$\rho_k: x_1 \mapsto \begin{pmatrix} \zeta_5^k & 0 \\ 0 & \zeta_5^{-k} \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} \zeta_5^{2k} & 0 \\ 0 & \zeta_5^{-2k} \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (k = 1, 2)$$

where $\zeta_5 = \exp(2\pi\sqrt{-1}/5)$. Then we have the same twisted Alexander polynomials $\Delta_{E_K}^{\alpha \otimes \rho_k}(t)$ and $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho_k}(t)$ for both $k = 1, 2$. They are expressed as, up to a factor $\pm t^n$ ($n \in \mathbb{Z}$),

$$\begin{aligned} \Delta_{E_K}^{\alpha \otimes \rho_k}(t) &= t^2 + 1, \\ \Delta_{E_K}^{\alpha \otimes Ad \circ \rho_k}(t) &= (t-1)(t^2 + 3t + 1) \\ &= (t-1)\Delta_K(-t) \end{aligned}$$

5.3. 5_1 knot ((2, 5)-torus knot). Let K be 5_1 knot as in the table of Rolfsen. This knot is the torus knot with type (2, 5). We consider the free Seifert surface illustrated as in Figure 3. The Lin presentation associated to this Seifert surface is expressed as

$$(11) \quad \pi_1(E_K) = \left\langle x_1, x_2, x_3, x_4, \mu \mid \begin{array}{ll} \mu x_1 \mu^{-1} = x_1 x_2^{-1}, & \mu x_2^{-1} x_1 \mu^{-1} = x_3 x_2^{-1}, \\ \mu x_2^{-1} x_3 \mu^{-1} = x_3 x_4^{-1}, & \mu x_4^{-1} x_3 \mu^{-1} = x_4^{-1} \end{array} \right\rangle.$$

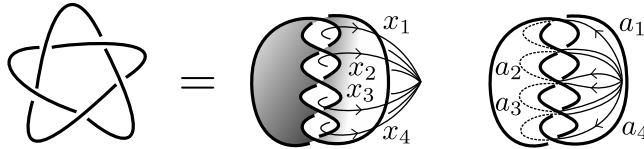


FIGURE 3. A free Seifert surface of 5_1 knot

The Alexander polynomial $\Delta_K(t)$ is given by $t^4 - t^3 + t^2 - t + 1$. The knot determinant is $\Delta_K(-1) = 5$. Since the number of conjugacy classes of irreducible metabelian representations is given by $(|\Delta_K(-1)| - 1)/2$, we have two irreducible metabelian representations, up

to conjugate. Proposition 2.8 and the relations in the Lin presentation (9), we have the following representatives ρ_1 and ρ_2 in those two conjugacy classes of irreducible metabelian representations:

$$\rho_k : x_i \mapsto \begin{pmatrix} (\zeta_5^k)^i & 0 \\ 0 & (\zeta_5^{-k})^i \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (k = 1, 2)$$

where $\zeta_5 = \exp(2\pi\sqrt{-1}/5)$. Then we have the following list of the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \rho_k}(t)$ and $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho_k}(t)$ for $k = 1, 2$:

	$\Delta_{E_K}^{\alpha \otimes Ad \circ \rho_k}(t)$	$\Delta_{E_K}^{\alpha \otimes \rho_k}(t)$
$k = 1$	$(t-1)\Delta_K(-t)(t^2 - \zeta_5)(t^2 - \zeta_5^{-1})$	$(t^2 + 1)(t^2 + \zeta_5^2)(t^2 + \zeta_5^{-2})$
$k = 2$	$(t-1)\Delta_K(-t)(t^2 - \zeta_5^2)(t^2 - \zeta_5^{-2})$	$(t^2 + 1)(t^2 + \zeta_5)(t^2 + \zeta_5^{-1})$

5.4. 5_2 knot. Let K be 5_2 knot in the table of Rolfsen. We consider the free Seifert surface illustrated as in Figure 4. The Lin presentation associated to this Seifert surface is expressed as

$$(12) \quad \pi_1(E_K) = \langle x_1, x_2, \mu \mid \mu x_1 \mu^{-1} = x_1 x_2^{-1}, \mu x_2^{-2} x_1 \mu^{-1} = x_2^{-2} \rangle.$$



FIGURE 4. A free Seifert surface of 5_2 knot

The Alexander polynomial $\Delta_K(t)$ is given by $2t^2 - 3t + 2$. The knot determinant is $\Delta_K(-1) = 7$. Since the number of conjugacy classes of irreducible metabelian representations is given by $(|\Delta_K(-1)| - 1)/2$, we have three irreducible metabelian representations, up to conjugate. By Proposition 2.8 and the relations in the Lin presentation (9), we have the following representatives ρ_1, ρ_2 and ρ_3 in those three conjugacy classes of irreducible metabelian representations:

$$\rho_k : x_1 \mapsto \begin{pmatrix} \zeta_7^k & 0 \\ 0 & \zeta_7^{-k} \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} \zeta_7^{2k} & 0 \\ 0 & \zeta_7^{-2k} \end{pmatrix}, \quad \mu \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (k = 1, 2, 3)$$

where $\zeta_7 = \exp(2\pi\sqrt{-1}/7)$. Then we have the following list of the twisted Alexander polynomial $\Delta_{E_K}^{\alpha \otimes \rho_k}(t)$ and $\Delta_{E_K}^{\alpha \otimes Ad \circ \rho_k}(t)$ for $k = 1, 2, 3$:

	$\Delta_{E_K}^{\alpha \otimes Ad \circ \rho_k}(t)$	$\Delta_{E_K}^{\alpha \otimes \rho_k}(t)$
$k = 1$	$(t-1)(\zeta_7^3 + \zeta_7^{-3} + 2)\Delta_K(-t)$	$(\zeta_7^2 + \zeta_7^{-2} + 2)(t^2 + 1)$
$k = 2$	$(t-1)(\zeta_7 + \zeta_7^{-1} + 2)\Delta_K(-t)$	$(\zeta_7^3 + \zeta_7^{-3} + 2)(t^2 + 1)$
$k = 3$	$(t-1)(\zeta_7^2 + \zeta_7^{-2} + 2)\Delta_K(-t)$	$(\zeta_7 + \zeta_7^{-1} + 2)(t^2 + 1)$

where $\zeta_7 = \exp(2\pi\sqrt{-1}/7)$.

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